Dada la matriz A

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

E imponiendo que sea simétrica, es decir: d=b,h=f y g=c

$$A = \begin{pmatrix} a & b & c \\ b & e & f \\ c & f & i \end{pmatrix}$$

Se requiere que además cumpla lo siguiente:

$$Cosa = (\varphi_1 \quad \varphi_2 \quad \varphi_3) \begin{pmatrix} a & b & c \\ b & e & f \\ c & f & i \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} = -6\varphi_1^2 - 6\varphi_2^2 - 6\varphi_3^2 - \sqrt{2}\varphi_2(\varphi_1 + \varphi_3)$$

Realizando la operación matricial:

$$(\varphi_1 \quad \varphi_2 \quad \varphi_3) \begin{pmatrix} a & b & c \\ b & e & f \\ c & f & i \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} = a\varphi_1^2 + e\varphi_2^2 + i\varphi_3^2 + 2b\varphi_1\varphi_2 + 2c\varphi_1\varphi_3 + 2f\varphi_2\varphi_3$$

Y por comparación de términos semejantes:

$$a\varphi_1^2 + e\varphi_2^2 + i\varphi_3^2 + 2b\varphi_1\varphi_2 + 2c\varphi_1\varphi_3 + 2f\varphi_2\varphi_3 = -6\varphi_1^2 - 6\varphi_2^2 - 6\varphi_3^2 - \sqrt{2}\varphi_2(\varphi_1 + \varphi_3)$$

$$a = -6; e = -6; i = -6; b = -\frac{1}{\sqrt{2}}; c = 0; f = -\frac{1}{\sqrt{2}}$$

Y por tanto:

$$A = \begin{pmatrix} -6 & -\frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & -6 & -\frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & -6 \end{pmatrix}$$

Para calcular los valores propios de A, resolvemos la identidad:

$$\det(A - \lambda I) = \begin{vmatrix} -6-\lambda & -\frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & -6-\lambda & -\frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & -6-\lambda \end{vmatrix} = (6+\lambda)(1-(6+\lambda)^2) = 0$$

$$\lambda_1 = -5; \lambda_2 = -6; \lambda_3 = -7$$

El cálculo de los vectores propios se realizará resolviendo la siguiente ecuación para cada uno de los valores propios hallados:

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda_i \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Para $\lambda = -5$

$$-6x - \frac{1}{\sqrt{2}}y = -5x \rightarrow y = -\sqrt{2}x$$

$$-\frac{1}{\sqrt{2}}x - 6y - \frac{1}{\sqrt{2}}z = -5y \rightarrow z = x$$

Para $\lambda = -6$

$$-6x - \frac{1}{\sqrt{2}}y = -6x \rightarrow y = 0$$

$$-\frac{1}{\sqrt{2}}x - 6y - \frac{1}{\sqrt{2}}z = -6y \rightarrow z = -x$$

Para $\lambda = -7$

$$-6x - \frac{1}{\sqrt{2}}y = -7x \rightarrow y = \sqrt{2}x$$

$$-\frac{1}{\sqrt{2}}x - 6y - \frac{1}{\sqrt{2}}z = -7y \rightarrow z = x$$

Tomando x=1 en todos los casos, obtenemos los siguientes vectores propios:

$$\overrightarrow{V_1} = \begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}; \overrightarrow{V_2} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}; \overrightarrow{V_3} = \begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

Calculamos sus módulos y normalizamos:

$$V_1 = \sqrt{(1)^2 + (-\sqrt{2})^2 + (1)^2} = 2;\ V_2 = \sqrt{(1)^2 + (0)^2 + (1)^2} = \sqrt{2};\ V_3 = \sqrt{(1)^2 + (\sqrt{2})^2 + (1)^2} = 2$$

$$\overrightarrow{V_1} = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}; \overrightarrow{V_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}; \overrightarrow{V_3} = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

Y con ellos la matriz M:

$$M = \begin{pmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{\sqrt{2}} & \frac{1}{2} \end{pmatrix}$$

Se desea ahora, que exista el siguiente cambio de variables:

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} = M \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$$

Resolviendo la identidad matricial:

$$\varphi_1 = \frac{1}{2}\psi_1 + \frac{1}{\sqrt{2}}\psi_2 + \frac{1}{2}\psi_3$$

$$\varphi_2 = -\frac{1}{\sqrt{2}}\psi_1 + \frac{1}{\sqrt{2}}\psi_3$$

$$\varphi_3 = \frac{1}{2}\psi_1 - \frac{1}{\sqrt{2}}\psi_2 + \frac{1}{2}\psi_3$$

Realizando el cambio de variable en "Cosa":

$$Cosa = -6\left(\tfrac{1}{2}\psi_1 + \tfrac{1}{\sqrt{2}}\psi_2 + \tfrac{1}{2}\psi_3\right)^2 - 6\left(-\tfrac{1}{\sqrt{2}}\psi_1 + \tfrac{1}{\sqrt{2}}\psi_3\right)^2 - 6\left(\tfrac{1}{2}\psi_1 - \tfrac{1}{\sqrt{2}}\psi_2 + \tfrac{1}{2}\psi_3\right)^2 - \sqrt{2}\left(-\tfrac{1}{\sqrt{2}}\psi_1 + \tfrac{1}{\sqrt{2}}\psi_3\right)\left(\left(\tfrac{1}{2}\psi_1 + \tfrac{1}{\sqrt{2}}\psi_2 + \tfrac{1}{2}\psi_3\right) + \left(\tfrac{1}{2}\psi_1 - \tfrac{1}{\sqrt{2}}\psi_2 + \tfrac{1}{2}\psi_3\right)\right)$$

Realizando algunas manipulaciones algebraicas:

$$Cosa = -6\left(\left(\tfrac{1}{2}\psi_1 + \tfrac{1}{2}\psi_3\right) + \tfrac{1}{\sqrt{2}}\psi_2\right)^2 - 6\left(-\tfrac{1}{\sqrt{2}}\psi_1 + \tfrac{1}{\sqrt{2}}\psi_3\right)^2 - 6\left(\left(\tfrac{1}{2}\psi_1 + \tfrac{1}{2}\psi_3\right) - \tfrac{1}{\sqrt{2}}\psi_2\right)^2 - (-\psi_1 + \psi_3)(\psi_1 + \psi_3) =$$

$$-6\left[\left(\tfrac{1}{2}\psi_1 + \tfrac{1}{2}\psi_3\right)^2 + \tfrac{1}{2}{\psi_2}^2 + \tfrac{1}{\sqrt{2}}\psi_2(\psi_1 + \psi_3)\right] - 6\left[\left(\tfrac{1}{2}\psi_1 + \tfrac{1}{2}\psi_3\right)^2 + \tfrac{1}{2}{\psi_2}^2 - \tfrac{1}{\sqrt{2}}\psi_2(\psi_1 + \psi_3)\right] - 3\left({\psi_1}^2 + {\psi_3}^2 - 2\psi_1\psi_3\right) - \left({\psi_3}^2 - {\psi_1}^2\right) =$$

$$-3\left[(\psi_1 + \psi_3)^2 + 2{\psi_2}^2\right] - 2{\psi_1}^2 - 4{\psi_3}^2 + 6\psi_1\psi_3 = -3\left({\psi_1}^2 + {\psi_3}^2 + 2\psi_1\psi_3 + 2{\psi_2}^2\right) - 2{\psi_1}^2 - 4{\psi_3}^2 + 6\psi_1\psi_3 =$$

$$Cosa = -5{\psi_1}^2 - 6{\psi_2}^2 - 7{\psi_3}^2 \Rightarrow \lambda_1{\psi_1}^2 + \lambda_2{\psi_2}^2 + \lambda_3{\psi_3}^2 \qquad c.q.d.$$